$$H = {}^1/_2\mu A_0 M_i M_i + {}^1/_2\mu (A_3 - A_0) M_i M_i \cos^2\theta + R_3 \cos\theta + R_2 \sin\theta, \quad A_0 = \begin{cases} A_1, & \mu = -1 \\ A_2, & \mu = 1 \end{cases} \tag{4.3}$$

From (1.2) it follows that if $E \geqslant 2G$, then $A_3 > A_1, A_3 > A_2$.

Calculating the derivatives $\partial H/\partial\theta, \partial^2 H/\partial\theta^2$, we conclude that a maximum of H is always reached at one point for all cases except a) $\mu = 1, R_3 = 0$ b) $\mu = -1, R_2 = 0$. Therefore, the solutions of the split and initial problems agree if there are no sections on the optimal rod on which conditions a) or b) are satisfied. For case a) an entire cone of values of $\mathbf{r}_3$ exists on which H reaches a maximum (Fig.3a). For case b) two values of the vector $\mathbf{r}_3$ exist on which H reaches a maximum (Fig.3b).

Example. Let us consider the case when $B_k = 0, p_i = m_i = 0$ and boundary conditions (1.7) for $\mathbf{P}_0 = 0$.

The optimal rod has the form 1 (Fig.4a) in the minimization problem $\Pi(l)$. Besides this solution, there is also a generalized solution 2 (Fig.4a). In addition to solutions with breaks, a smooth optimal solution can also be constructed. Condition a is realized for these optimal solutions.

The optimal rod has the form 1 (Fig.4b) for the maximization problem $\Pi(l)$. In addition to this solution, there is also the generalized solution 2 (Fig.4b). Condition b is realized for these optimal solutions.

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ON SYMMETRIC AND NON-SYMMETRIC CONTACT PROBLEMS OF THE THEORY OF ELASTICITY*

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Contact problems of the theory of elasticity can be subdivided into two major classes: symmetric contact problems for which the kernel of integral equations of the convolution type are even or odd functions, and non-symmetric contact problems for which the kernels are given by the sum of odd and even functions, Certain problems from this latter class were apparently examined first in /1–3/. In this paper a general approach to their study is given and an approximate solution is constructed; the results are demonstrated in two new problems.

1. As is well-known /4–6/, may plane and axisymmetric contact problems of the theory of elasticity reduce to determining the contact forces from an integral equation of the first kind with a different kernel of the form

$$\int_{-1}^{1} \varphi(\xi)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x) \quad (|x| \leqslant 1) \tag{1.1}$$

$$k(t) = \frac{1}{2} \int_{-\infty+ic}^{\infty+ic} K(\zeta)\, e^{i\zeta t}\, d\zeta \tag{1.2}$$

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Here $\varphi(x)$ is the dimensionless contact force, λ is the dimensionless geometric or physicomechanical parameter, $f(x)$ is a function governed by the contact condition, and $K(\zeta)$ is an analytic function of the complex variable $\zeta = u + iv$ which is even and real on the real axis.

The majority of static contact problems-encountered for the behaviour of the symbol $K(\zeta)$ of the kernel (1.2) at zero and infinity on the real axis can be divided into three groups

$$K(u) \to |u|^{-1} \; (|u| \to \infty), \quad K(u) \to A \; (u \to 0) \tag{1.3}$$

$$K(u) \to |u|^{-1} \; (|u| \to \infty), \quad K(u) \to B\,|u|^{-1} \; (u \to 0) \tag{1.4}$$

$$K(u) \to |u|^{-1} \; (|u| \to \infty), \quad K(u) \to Cu^{-2} \; (u \to 0) \tag{1.5}$$

where A, B and C are constants. Cases (1.3) and (1.4) have been studied in detail /4, 7/. We will investigate case (1.5) in detail in this paper.

Let us consider the auxiliary integral

$$J_\varepsilon(t) = \int_\Gamma \frac{H(\zeta)}{\zeta^2 - \varepsilon^2} e^{i\zeta t}\, d\zeta \quad (\varepsilon > 0) \tag{1.6}$$

over the contour Γ in the plane of the complex variable ζ, where $H(\zeta)$ is an analytic function increasing as $|\zeta|$ on a regular system of contours $C_n \subset C_{n+1}$ as $n \to \infty$ /4, 8/, and is even and real on the real axis.

Let the contour Γ have the form shown in Fig.1 $(c > 0)$. Then assuming that the function $H(\zeta)$ is regular within the domain outlined by the contour Γ, and using the theory of residues we find

$$J_\varepsilon(t) = \pi\varepsilon^{-1} H(\varepsilon) \sin \varepsilon t \tag{1.7}$$

On the other hand, letting l tend to infinity and remarking that the integrals over the segments AB and CD vanish, we will have

$$J_\varepsilon(t) = \int_{-\infty+ic}^{\infty+ic} \frac{H(\zeta)}{\zeta^2 - \varepsilon^2} e^{i\zeta t}\, d\zeta - \int_{-\infty}^{\infty} \frac{H(u)}{u^2 - \varepsilon^2} e^{iut}\, du \tag{1.8}$$

On the basis of (1.7) and (1.8) we obtain as $\varepsilon \to 0$

$$k(t) = \int_0^\infty K(u) \cos ut\, du + \frac{\pi}{2} H(0)\, t \tag{1.9}$$

where it is assumed that $H(u) = u^2 K(u)$. Analogously, if $c < 0$, then

$$k(t) = \int_0^\infty K(u) \cos ut\, du - \frac{\pi}{2} H(0)\, t \tag{1.10}$$

Now, let the contour Γ in (1.6) coincide with the real axis $(c = 0)$. Then we will understand the integral (1.6) in the Cauchy principle-value sense. Here, letting ε tend to zero, we obtain

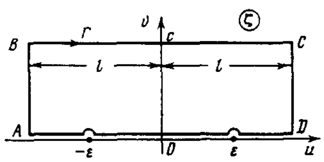

Fig.1

$$k(t) = \int_0^\infty K(u) \cos ut\, du \tag{1.11}$$

We note that the integral in (1.9)−(1.11) is defined to within an infinite constant D that can be extracted as follows, say:

$$\int_0^\infty K(u) \cos ut\, du = \int_0^\infty K(u) (\cos ut - e^{-u^2})\, du + D \tag{1.12}$$

The presence of this constant in the kernel $k(t)$ does not enable a connection between the rigid translational displacement of the stamp and the force acting on it to be determined in problems of group (1.5) (as also in problems of group (1.4), however).

2. Taking account of (1.9) and (1.10), the integral equation (1.1) can be written in the form

$$\int_{-1}^{1} \varphi(\xi) \left[\int_0^\infty K(u) \cos u \frac{\xi - x}{\lambda}\, du \pm \frac{\pi}{2} H(0) \frac{\xi - x}{\lambda} \right] d\xi = \pi f(x) \tag{2.1}$$

$$(|x| \leqslant 1)$$

Here and henceforth, in the case of a double sign, the upper sign corresponds to $c > 0$, and the lower sign to $c < 0$ in (1.2).

After differentiation with respect to x we obtain from (2.1)

$$F(x) = \pi\lambda f'(x) \ (|x| \leqslant 1) \tag{2.2}$$

$$\left(F(x) = \int_{-1}^{1} \varphi(\xi) \left[\int_0^\infty \frac{H(u)}{u} \sin u \frac{\xi - x}{\lambda} du \mp \frac{\pi}{2} H(0)\right] d\xi\right)$$

In (2.2), and later for $c = 0$, the component with the double sign should be omitted. We introduce the notation

$$f_*'(x) = (\pi\lambda)^{-1} F(x) \ (|x| > 1) \tag{2.3}$$

It can be established on the basis of Theorem 5 in Sect.3 of Ch. 1 in /9/ that

$$\int_0^\infty \frac{H(u)}{u} \sin ut\, du \to \begin{cases} {}^1/_2\pi H(0), & t \to +\infty \\ -{}^1/_2\pi H(0), & t \to -\infty \end{cases} \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$f_*'(x) \to \begin{cases} {}^1/_2 P\lambda^{-1} H(0)(1 \mp 1), & x \to -\infty \\ -{}^1/_2 P\lambda^{-1} H(0)(1 \pm 1), & x \to +\infty \end{cases} \tag{2.5}$$

where the constant P is considered known and defined by the integral

$$P = \int_{-1}^{1} \varphi(x)\, dx \tag{2.6}$$

Therefore, the position of the contour of integration in (1.2) can be determined in conformity with (2.5) by the conditions at infinity for the functions $f_*'(x)$.

The effective solution of (2.2) can be obtained by the asymptotic methods of "large and small λ " /4, 10/.

We will limit ourselves to constructing the approximate solution of the integral equation (2.2) based on approximating the function $H(u)$ by the expression

$$H(u) = u \operatorname{cth} Au \ (H(0) = A^{-1}) \tag{2.7}$$

The approximation (2.7) corresponds to the fundamental properties (1.5) of the function $K(u) = u^{-2} H(u)$. By using (2.6) and (2.7) and the generalized value of the integral

$$\int_0^\infty \operatorname{cth} Au \sin ut\, du = \frac{\pi}{2A} \operatorname{cth} \frac{\pi t}{2A} \tag{2.8}$$

we can rewrite (2.2) in the form

$$\int_{-1}^{1} \varphi(\xi) \operatorname{cth} \frac{\gamma(\xi - x)}{2} d\xi = 2A\lambda f'(x) \pm P \quad \left(|x| \leqslant 1, \ \gamma = \frac{\pi}{A\lambda}\right) \tag{2.9}$$

By introducing new variables and notation by means of the formulas

$$\rho = e^{\gamma\xi}, \ r = e^{\gamma x}, \ \varphi(\xi) \equiv \psi(\rho) \tag{2.10}$$

$$f'(x) \equiv u(r), \ \alpha = e^{-\gamma}, \ \beta = e^{\gamma}$$

(2.9) is converted to a singular integral equation whose inversion formula is known /11/. Applying this formula and then returning to the original variables and notation of (2.10), we obtain

$$\varphi(x) = \frac{1}{\pi\Delta(x)} \left[Q - \frac{P\gamma}{2}(1 \pm 1)(\operatorname{ch}\gamma - e^{\gamma x}) - \right. \tag{2.11}$$

$$\left. \gamma \int_{-1}^{1} \frac{e^{\gamma\xi} f'(\xi)\Delta(\xi)}{e^{\gamma\xi} - e^{\gamma x}} d\xi\right], \ \ \Delta(x) = [(\beta - e^{\gamma x})(e^{\gamma x} - \alpha)]^{1/2}$$

On the basis of (2.6) and (2.11) the constant Q can be expressed in terms of P.

For instance, let $f(x) = f = \text{const}$. In this case

$$\varphi(x)=\frac{P}{A\lambda\Delta(x)}\left[1-\frac{1\pm1}{2}(1-e^{\gamma x})\right] \tag{2.12}$$

It can be shown that the accuracy of the approximate solution (2.12) of the integral equation (2.2) is not less than the accuracy of approximation (2.7).

3. We will examine two problems as examples.

The problem of the shear of an elastic layer by a strip stamp. An elastic layer of thickness h occupies the domain $-h\leqslant y\leqslant 0$, $|x|<\infty$, $|z|<\infty$, where x, y, z are rectangular Cartesian coordinates. The upper face of the layer is fastened to a rigid strip (stamp) of infinite length of constant width $2a$. The domain of strip contact with the layer is determined by the conditions $y=0$, $|x|\leqslant a$, $|z|<\infty$. The upper face of the layer outside the domain of contact and the lower face are load-free (it can also be considered that the lower face is supported without friction on a rigid foundation). The layer is fastened at infinity for $x=+\infty$ (Fig.2), or for $x=-\infty$, of for $x=\pm\infty$ (Fig.3). The strip is shifted in the direction of the z axis by a force T referred to units of the strip length.

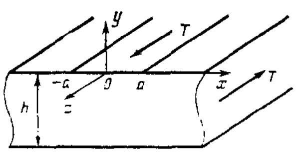

Fig.2

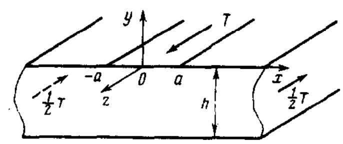

Fig.3

In the case of pure shear, the function w characterizing the projection of the displacement vector on the z axis satisfies the Laplace equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0 \tag{3.1}$$

We shall use the solution of equation (3.1) in the form of a generalized Fourier transform

$$w(x,y)=\frac{1}{2\pi}\int\limits_{-\infty+ic}^{\infty+ic} D(\zeta,y)\,e^{-i\zeta x}\,d\zeta,\quad D(\zeta,y)=D_1(\zeta)\operatorname{sh}\zeta y+D_2(\zeta)\operatorname{ch}\zeta y \tag{3.2}$$

By using Hooke's law, we find from (3.2) (μ is the shear modulus)

$$\tau_{xz}(x,y)=-\frac{i\mu}{2\pi}\int\limits_{-\infty+ic}^{\infty+ic}\zeta D(\zeta,y)\,e^{-i\zeta x}\,d\zeta \tag{3.3}$$

$$\tau_{yz}(x,y)=\frac{\mu}{2\pi}\int\limits_{-\infty+ic}^{\infty+ic}\zeta\,[D_1(\zeta)\operatorname{ch}\zeta y+D_2(\zeta)\operatorname{sh}\zeta y]\,e^{-i\zeta x}\,d\zeta \tag{3.4}$$

We represent the function $\tau_{yz}(x,0)$ in the form

$$\tau_{yz}(x,0)\equiv\tau(x)=\frac{1}{2\pi}\int\limits_{-\infty+ic}^{\infty+ic}\tau_*(\zeta)\,e^{-i\zeta x}\,d\zeta \tag{3.5}$$

Taking account of the representations (3.4) and (3.5) and the boundary condition $\tau_{yz}(0,-h)=0$, we express the functions D_1 and D_2 in terms of τ_*

$$D_1(\zeta)=(\mu\zeta)^{-1}\tau_*(\zeta),\quad D_2(\zeta)=(\mu\zeta)^{-1}\operatorname{cth}\zeta h\,\tau_*(\zeta) \tag{3.6}$$

Then using the procedure developed in Sect.1, and the Fourier transform inversion formula

$$\tau_*(\zeta)=\int\limits_{-a}^{a}\tau(\xi)\,e^{i\xi x}\,d\xi$$

we determine from (3.2), (3.3) and (3.6)

$$w(x,y)=\frac{1}{\pi\mu}\int\limits_{-a}^{a}\tau(\xi)\left[\int\limits_{0}^{\infty}\frac{U_1(u,y)}{u}\cos u\frac{\xi-x}{h}\,du\pm\frac{\pi}{2}\,\frac{\xi-x}{h}\right]d\xi \tag{3.7}$$

$$\tau_{xz}(x, y) = \frac{1}{\pi h} \int_{a}^{a} \tau(\xi) \left[\int_{0}^{\infty} U_1(u, y) \sin u \frac{\xi - x}{h} du \mp \frac{\pi}{2} \right] d\xi \tag{3.8}$$

Here

$$U_1(u, y) = \operatorname{sh} \frac{uy}{h} + \operatorname{cth} u \operatorname{ch} \frac{uy}{h}, \quad \mathrm{T} = \int_{-a}^{a} \tau(x)\, dx \tag{3.9}$$

After changing to dimensionless variables by means of the formulas $x = ax'$ and $\xi = a\xi'$, an integral equation for the problem under investigation can be obtained from (3.7) in the form (1.1), (1.2), (2.1), (2.2), where

$$\tau(x) \equiv \varphi(x'), \ K(u) = u^{-1} \operatorname{cth} u, \ H(0) = 1$$
$$f(x') = \mu\varepsilon/a = \text{const}, \ \lambda = h/a \tag{3.10}$$

and ε is the constant displacement of points of the contact domain (the primes are omitted in the formulas mentioned). From (3.8) we determine the force T_* to which the stresses τ_{xz} in any section of the layer of height h and unit width orthogonal to the x axis are statically equivalent, where

$$\mathrm{T}_*(x) = \int_{-h}^{0} \tau_{xz}\, dy \tag{3.11}$$

Substituting τ_{xz} in the form of (3.8) into (3.11), we obtain after reduction

$$\mathrm{T}_*(x) = \frac{1}{2} \int_{-a}^{a} \tau(\xi) \operatorname{sign}(\xi - x)\, d\xi \mp \frac{1}{2} \mathrm{T} \tag{3.12}$$

It follows from (3.12) that

$$\mathrm{T}_* = \begin{cases} -\mathrm{T}, & c > 0 \\ 0, & c < 0 \\ -{}^1/_2\mathrm{T}, & c = 0 \end{cases} \quad \text{when} \quad x > a \tag{3.13}$$
$$\mathrm{T}_* = \begin{cases} 0, & c > 0 \\ \mathrm{T}, & c < 0 \\ {}^1/_2\mathrm{T}, & c = 0 \end{cases} \quad \text{when} \quad x < -a$$

Hence it follows that the case of the contour of integration in (1.2) being above the real case $c > 0$ (below the real axis $c < 0$) corresponds to fastening the layer for $x = +\infty$ $(x = -\infty)$. The case of coincidence of the contour of integration in (1.2) with the real axis $(c = 0)$ corresponds to symmetric fastening of the layer for $x = \pm\infty$.

For the case under consideration, $\varphi(x)$ in the form (2.12) is the exact solution of integral equation (2.2) for $A = 1$. Taking account of (2.6), (3.9) and (3.10), from (2.12) we obtain an expression determining the stress τ_{yz} in the contact domain between a strip and a layer in the form

$$\tau(x) = \frac{\mathrm{T}}{h\Delta(x/a)} \left[1 - \frac{1 \pm 1}{2} (1 - e^{\gamma x/a}) \right] \quad \left(\gamma = \frac{\pi a}{h} \right) \tag{3.14}$$

The problem of the torsion of a cylinder by a rigid belt. Let an elastic infinite circular cylinder occupy the domain $0 \leqslant r \leqslant R, 0 \leqslant \theta < 2\pi, |z| < \infty$, where r, θ, z are cylindrical coordinates, and R is the radius of the cylinder. A rigid belt is set on the cylinder without tension, and the cylinder and belt are interconnected. The contact domain of the belt and cylinder is defined by the conditions $|z| \leqslant a, 0 \leqslant \theta < 2\pi, r = R$. The cylinder surface outside the domain of contact with the belt is load-free. The cylinder is fixed at infinity for $z = +\infty$ or for $z = -\infty$, or for $z = \pm\infty$. A torque of magnitude M is applied to the belt. The moment M is related to the tangential stress $\tau(z) \equiv \tau_{r\theta}(R, z)$ $(|z| \leqslant a)$ that occur in the contact domain between the belt and the cylinder, by the formula

$$M = 2\pi R^2 \mathrm{T}, \quad \mathrm{T} = \int_{-a}^{a} \tau(z)\, dz \tag{3.15}$$

The following formulas, relating the projection v of the displacement vector on the θ axis and the stress $\tau_{\theta z}$ to the unknown contact stresses $\tau(z)$ can be obtained by using the Fourier integral transform:

$$v(r, z) = \frac{1}{\pi\mu} \int_{-a}^{a} \tau(\xi) \left[\int_{0}^{\infty} U_2(u, r) \cos u \frac{\xi - z}{R} du \pm 2\pi \frac{r}{R} \frac{\xi - z}{R} \right] d\xi \tag{3.16}$$

$$\tau_{\theta z}(r, z) = \frac{1}{\pi R} \int_{-a}^{a} \tau(\xi) \left[\int_{0}^{\infty} u U_2(u, r) \sin u \frac{\xi - z}{R} du \mp 2\pi \frac{r}{R} \right] d\xi \tag{3.17}$$

$$U_2(u, r) = I_1(ur/R) [u I_0(u) - 2I_1(u)]^{-1}$$

Here $I_n(z)$ is the cylindrical function of imaginary argument.

Changing to dimensionless variables by means of the formulas $z = ax$, $\xi = a\xi'$ the integral equation for the problem under consideration can be obtained from (3.16) in the form (1.1), (1.2), (2.1), (2.2), where

$$\tau(\xi) \equiv \varphi(\xi'), \ f(x) = \mu\lambda\varepsilon = \text{const}, \ \lambda = R/a \tag{3.18}$$

$$K(u) = U_2(u, R), \ H(0) = 4$$

ε is the angle of belt rotation (the primes are omitted in the formulas mentioned). Let M_* be the torque originating in any section of the cylinder orthogonal to its axis. In this case

$$M_*(z) = 2\pi \int_0^R r^2 \tau_{\theta z}(r, z) dr \tag{3.19}$$

Inserting $\tau_{\theta z}$ in the form (3.17) into (3.19), we obtain after evaluating the integrals

$$M_*(z) = \pi R^2 \int_{-a}^{a} \tau(\xi) [\text{sign}(\xi - z) \mp 1] d\xi \tag{3.20}$$

In particular, relationships analogous to (3.13) follow from (3.20) (on replacing T_* by M_*, T by $2\pi R^2 T$, and x by z), as does also the deduction relative to the location of the contour of integration in (1.2), which is analogous to that made when investigating the problem of the shear of an elastic layer.

In this case $\varphi(x)$ in the form (2.12) is an approximate solution of integral equation (2.2). Here $A = H^{-1}(0) = 1/4$ by virtue of (3.18), and the error in approximation (2.7) does not exceed 16% for $0 \leqslant u < \infty$. From (2.12), (3.15) and (3.18) we obtain an approximate expression to determine the tangential stresses $\tau(z)$ in the contact domain between the belt and the cylinder in the form

$$\tau(z) = \frac{2M}{\pi R^3 \Delta(z/a)} \left[1 - \frac{1 \pm 1}{2} (1 - e^{\gamma z/a}) \right] \left(\gamma = \frac{4\pi a}{R} \right)$$

We note that the problem of the tension of an elastic strip with a free lower face (or resting without friction on a rigid foundation), a tension-rigid but absolutely flexible cover-plate welded to the upper face, as well as the problem of the tension of a tension-rigid cylinder but with an absolutely flexible annular cover-plate welded to its surface can be considered in the same way.

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